OPTIMAL STRATEGIES IN GAMES OF SURVIVAL

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RM-777

18 February 1952

Report Documentation Page

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1. REPORT DATE **18 FEB 1952**	2. REPORT TYPE	3. DATES COVERED **00-00-1952 to 00-00-1952**

4. TITLE AND SUBTITLE **Optimal Strategies in Games of Survival**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Rand Corporation,Project Air Force,1776 Main Street, PO Box 2138,Santa Monica,CA,90407-2138**	8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**8**	

SUMMARY

A simple procedure is given to calculate
the optimal strategies and the survival prob-
ability for a general class of games of survival.
Theorem 1 shows that for games of this particular
type, the optimal strategy for both players de-
pends only on the amount of money held by the
second player.

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In RM-776, the author considered games of the following
type. Players I and II play a zero-sum, 2 x 2 game con-
tinually, play ceasing when one of the players runs out of
money. If the payoff matrix to I is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad ; \quad a_{11}, a_{22} > 0; \; a_{12}, a_{21} < 0,$$

then the above game was shown to have a solution. The play for
I's survival and II's ruin are equivalent for player I. Player
I can assume himself a probability $v_n(k)$ of survival (n is
the amount of money in the game, k is I's initial share of
the money) if he plays optimally, while II can limit him to
this probability of survival.

The difficulty, in practice, lies in the calculation of
$v_n(k)$. (The optimal mixed strategies are simple functions of
$v_n(k)$.) We give a case where some general statements may be

made about the solution and where the computation is greatly
shortened. We shall always assume that n, k, and a_{ij} are
integers.

Lemma 1: Suppose that I cannot lose more than 1 unit
of money; i.e. the payoff matrix is of the form

(1) $$A = \begin{pmatrix} a & -1 \\ -1 & b \end{pmatrix}; \quad a, b > 0.$$

then,

(2) $$v_{n+1}(k+1) = v_n(k) + \left[1-v_n(k)\right] v_{n+1}(1).$$

Proof: Let II play his optimal strategy against the
following strategy of I's: I reserves one unit of money and
plays his optimal strategy for the reduced game, where his
money and the total amount of money in the game is reduced by
one. If I has only one unit, he plays arbitrarily. We
calculate I's probability of survival if both players play
in the above fashion.

First, it is known that the payoff is independent of a
players strategy provided one of the players uses his optimal
strategy. Assume that I has k+1 units and II has n-k.
Since II plays his optimal strategy, the payoff (I's probability
of survival) is $v_{n+1}(k+1)$. On the other hand, I can win in
two mutually exclusive ways: he never falls down to one unit,
or he does but comes back to win. The first case occurs with
probability $v_n(k)$ since I plays his optimal strategy in the
reduced game. The second case occurs with probability $1-v_n(k)$
of falling down to one unit, and the probability $v_{n+1}(1)$ of

then surviving (since II plays optimally in the (n+1) game). Hence, I's probability of survival is

$$v_n(k) + \left[1-v_n(k)\right] v_{n+1}(1).$$

thus,

$$(2) \qquad v_{n+1}(k+1) = v_n(k) + \left[1-v_n(k)\right] v_{n+1}(1),$$

proving the lemma.

Theorem 1: For a game of the type (1), I's optimal strategy is to "reserve" one unit of money and play optimally in the reduced game.

By induction then, if I has k+1 unit of money and II has n, I's optimal strategy is to reserve k units of money and play optimally in the game in which he has 1 unit and II has n. Hence, I's optimal strategy depends only on the amount of money that II holds.

Proof: Let I have k+1 units and let there be n+1 units in the game. Suppose I plays his (unique) optimal strategy against any strategy of II's. I's strategy and II's strategy induce strategies for the reduced (n,k) game. Let p be the probability of I winning the reduced game (i.e. of his winning the original game without ever being reduced to one unit). Since I's strategy is optimal,

$$v_{n+1}(k+1) = p + (1-p) v_{n+1}(1).$$

By (2), $p = v_n(k)$. This is so for every one of II's strategies. Hence, I's reduced strategy is optimal in the reduced game, proving the theorems.

With the help of formula (4) of RM-776, we have

$$v_{n+1}(1) = \frac{v_{n+1}(1+a) \ v_{n+1}(1+b)}{v_{n+1}(1+a) + v_{n+1}(1+b)} \ .$$

Using formula (2), and substituting in this equation, we obtain a quadratic equation for $v_{n+1}(1)$, having only one positive root. The solution is:

$$(3) \quad v_{n+1}(1) = \frac{\sqrt{v_n(a) \ v_n(b)}}{1+ \sqrt{v_n(a) \ v_n(b)}} \ .$$

Using the trivial

$$(4) \quad v_2(1) = \frac{1}{2} \ ,$$

it is possible to calculate $v_n(k)$ in general by induction on n with the help of (4), (3), and (2). The optimal strategies are then easily obtained. Letting $p_n(k)$ be the probability of I playing row 1 in his optimal strategy, we have by formula (3) of RM-776,

$$(5) \quad p_n(1) = \frac{v_n(1+b)}{v_n(1+a) + v_n(1+b)} = \frac{v_n(1)}{v_n(1+a)}$$

This is also II's probability of playing column 1 in his optimal strategy when I has k units and II has n-k. By theorem 1,

(6) $\quad p_{n+1}(k+1) = p_n(k) = p_{n-k+1}(1),$

so that the probabilities $p_n(1)$ give all the optimal strategies which may occur.

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